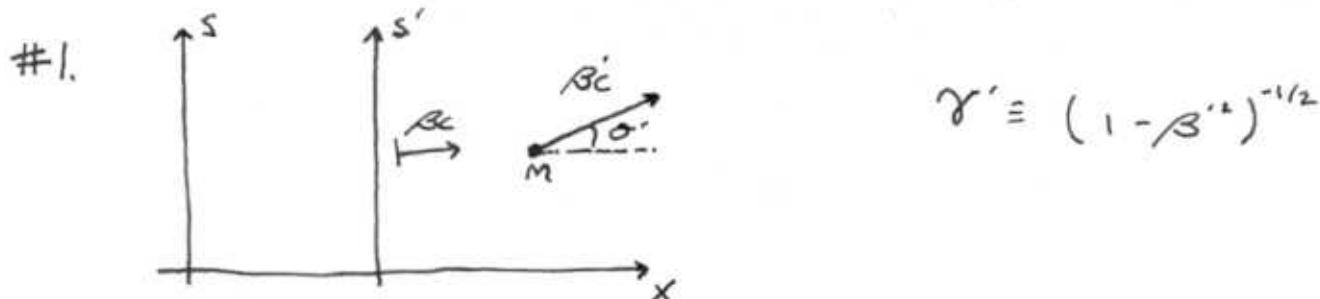


Physics 110B
Homework #6



$$\begin{aligned} \mathbf{p}' &= (mc\gamma', m\beta'c\gamma'\cos\theta', m\beta'c\gamma'\sin\theta', 0) \\ &= (p'_0, p'_1, p'_2, p'_3) \end{aligned}$$

$$\mathbf{p} = (p_0, p_1, p_2, p_3) = (p_0, p_1, p_2, p_3)$$

Now, using a Lorentz transformation:

$$p_0 = \gamma(p'_0 + \beta p'_1) \quad ; \quad p_2 = p'_2$$

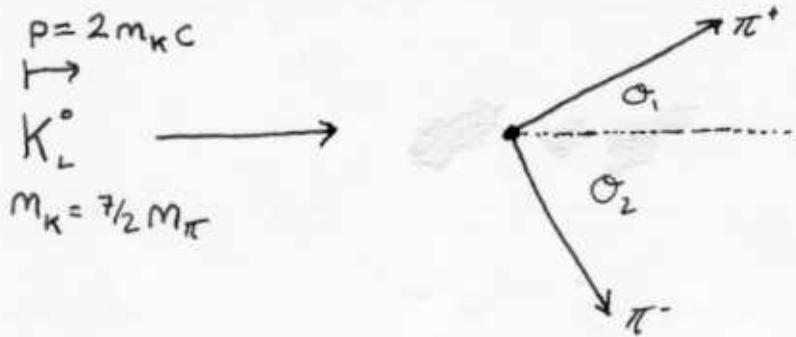
$$p_1 = \gamma(p'_1 + \beta p'_0) \quad ; \quad p_3 = p'_3$$

So,

$$\begin{aligned} \tan\theta &= \frac{\sin\theta}{\cos\theta} = \frac{p_3 \sin\theta}{p_2 \cos\theta} = \frac{p_2}{p_1} = \frac{p'_2}{\gamma(p'_1 + \beta p'_0)} \\ &= \frac{\gamma' m \beta' c \sin\theta'}{\gamma(m \beta' c \gamma' \cos\theta' + \beta m c \gamma')} \end{aligned}$$

$\tan\theta = \frac{\sin\theta'}{\gamma(\cos\theta' + \beta/\beta')}$

#2.
and #3.



In the rest frame the Kaon decays isotropically, therefore the flux of decay pions per unit solid angle is constant:

$$\frac{dN}{d\Omega' dt} = \text{Constant} \quad (\text{Since, isotropic})$$

In the lab frame the flux of decay pions per unit solid angle is:

$$\frac{dN}{d\Omega dt} = \underbrace{\frac{dN}{d\Omega' dt}}_{\text{const.}} \cdot \frac{d\Omega'}{d\Omega} = \text{const.} \frac{\sin\alpha' d\alpha' d\phi'}{\sin\alpha d\alpha d\phi}$$

Since ϕ is perpendicular to the direction of motion it remains the same both in the Lab and Rest Frames: $\phi = \phi'$.

$$\text{Thus, } \frac{dN}{d\Omega dt} = \text{const.} \frac{\sin\alpha'}{\sin\alpha} \frac{d\alpha'}{d\alpha}$$

When $\sin\alpha = 0$ ($\alpha = 0, \pi$) the flux per solid angle is infinite, however, these are the trivial solutions. The flux per solid angle will also go to infinity when $d\alpha'/d\alpha \rightarrow \infty$:

$$\frac{dN}{d\Omega dt} \rightarrow \infty \quad \text{when} \quad \frac{d\alpha'}{d\alpha} \rightarrow \infty$$

$$\Rightarrow \text{Or when} \quad \frac{d\alpha}{d\alpha'} \rightarrow 0$$

So, we need to find when $\frac{d\alpha}{d\alpha'} = 0$

$$\alpha = \arctan \left(\frac{\sin\alpha'}{\gamma(\cos\alpha' + (\beta/\beta'))} \right) \quad (\text{from problem #1})$$

$$\frac{d\theta'}{d\alpha'} = \frac{d}{d\alpha'} (\arctan x) = \frac{dx}{d\alpha'} \frac{d}{dx} (\arctan x) = 0$$

$$\Rightarrow \frac{dx}{d\alpha'} = 0$$

$$\Rightarrow \frac{d}{d\alpha'} \left(\frac{\sin\alpha'}{\gamma(\cos\alpha' + \beta/\beta')} \right) = \frac{\cos\alpha'(\cos\alpha' + \beta/\beta') - \sin\alpha'(-\sin\alpha')}{\gamma(\cos\alpha' + \beta/\beta')^2} = 0$$

$$\Rightarrow \cos\alpha'(\cos\alpha' + \beta/\beta') + \sin^2\alpha' = 0$$

$$\cos^2\alpha' + \sin^2\alpha' + (\cos\alpha')\beta/\beta' = 0$$

$$(\cos\alpha')\beta/\beta' = -1$$

$$\boxed{\arccos(-\beta'/\beta) = \alpha'}$$

Thus, we need to find β and β' :

$$p_{K_0} = (E_K/c, 2m_K c, 0, 0)$$

$$p_K \cdot p_K = E_K^2 - 4m_K^2c^2 = m_K^2c^2 \Rightarrow E_K = \sqrt{5}m_Kc$$

$$\text{Since, } p_K = \left(\frac{E_K}{c}, \vec{p}_K\right) = (\gamma m_K c, \gamma \vec{\beta} c m_K)$$

$$\text{Thus, } \frac{E_K/c}{p_K} = \frac{\gamma m_K c}{\gamma \beta c m_K} = \frac{1}{\beta}$$

So,

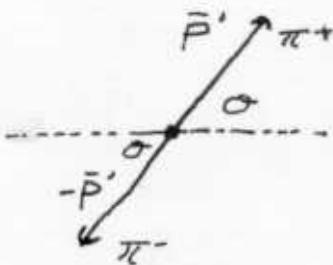
$$\beta = \frac{p_K}{E_K} = \frac{2m_K c}{\sqrt{5}m_K c} = \boxed{\frac{2}{\sqrt{5}}} = \beta$$

$$p_K = \gamma \beta c m_K = 2m_K c$$

$$\gamma \beta = 2 \Rightarrow \boxed{\gamma = \sqrt{5}}$$

In the Rest Frame,

$$\textcircled{M_K} \longrightarrow E'_K = m_K c^2$$



From energy conservation:

$$E'_K = E'_{\pi^+} + E'_{\pi^-}$$

$$m_K c^2 = \sqrt{c^2 p'^2 + m_{\pi}^2 c^4} + \sqrt{p'^2 c^2 + m_{\pi}^2 c^4}$$

$$\left(\frac{7}{4} m_{\pi} c^2\right)^2 = c^2 p'^2 + m_{\pi}^2 c^4$$

$$\frac{33}{4 \cdot 4} m_{\pi}^2 = p'^2 / c^2 \Rightarrow p'_{\pi} = \frac{\sqrt{33}}{4} m_{\pi} c$$

$$E'_{\pi} = E'_K / 2 = m_K c^2 / 2 = \frac{7}{4} m_{\pi} c^2$$

$$\text{Thus, } \Rightarrow \beta' = \frac{p'_{\pi} c}{E'_{\pi}} = \frac{\sqrt{33}/4 m_{\pi} c^2}{7/4 m_{\pi} c^2} = \boxed{\frac{\sqrt{33}}{7}} = \beta'$$

$$\Rightarrow \sigma' = \arccos\left(-\frac{\beta'}{\beta}\right) = \arccos\left(-\frac{2}{\sqrt{5}} \frac{7}{\sqrt{33}}\right) = 156^\circ$$

$$\Rightarrow \phi = \arctan\left(\frac{\sin \sigma'}{\gamma(\cos \sigma' + \beta/\beta')}\right) = \arctan\left(\frac{\sin 156}{\sqrt{5}(\cos 156 + \frac{2 \cdot 7}{\sqrt{6 \cdot 33}})}\right)$$

$$\boxed{\phi = 45.9^\circ}$$

the Flux per unit solid angle goes to infinity at this angle

$$\#4. \quad A^\mu = \{V_c, \bar{A}\}, \quad k^\mu = \{\omega/c, \bar{k}\}$$

$$J^\mu = \{c\rho, \bar{J}\}, \quad \partial^\mu = \left\{ \frac{\partial}{c\partial t}, -\bar{\nabla} \right\}$$

$$P^\mu = \{E/c, \bar{P}\}$$

(a) Generalized de Broglie relation:

$$P = h/\lambda = 2\pi\hbar/\lambda = \hbar k, \quad E = \hbar\omega$$

$$\Rightarrow \left(\frac{E}{c}, \bar{P} \right) = (\omega/c, \bar{k})\hbar \Rightarrow \boxed{P^\mu = \hbar k^\mu}$$

(b) Conservation of electric charge:

$$\frac{\partial \rho}{\partial t} + \bar{\nabla} \cdot \bar{J} = 0 \Rightarrow \left(\frac{\partial}{c\partial t}, -\bar{\nabla} \right) \cdot (c\rho, -J) = 0$$

$$\Rightarrow \boxed{\partial_\mu J^\mu = 0}$$

(c) Lorentz Gauge condition:

$$\bar{\nabla} \cdot \bar{A} + \frac{1}{c^2} \frac{\partial V}{\partial t} = 0 \Rightarrow \left(\frac{\partial}{c\partial t}, \bar{\nabla} \right) \cdot \left(\frac{V}{c}, -\bar{A} \right) = 0$$

$$\Rightarrow \boxed{\partial_\mu A^\mu = 0}$$

(d) the Wave Equation, including sources, for the electromagnetic potentials:

$$\nabla^2 V - \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon_0}, \quad \nabla^2 \bar{A} - \frac{1}{c^2} \frac{\partial^2 \bar{A}}{\partial t^2} = -\mu_0 \bar{J}$$

$$\Rightarrow \partial_\mu \partial^\mu = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \Rightarrow \partial_\mu \partial^\mu (V_c, \bar{A}) = (\frac{\rho}{\epsilon_0}, \mu_0 \bar{J})$$

$$\Rightarrow \partial_\mu \partial^\mu A^\nu = \mu_0 \left(\frac{\rho}{\epsilon_0 \mu_0}, \bar{J} \right) \Rightarrow \boxed{\partial_\mu \partial^\mu A^\nu = \mu_0 J^\nu}$$

$$\#5. \quad \bullet a'^\mu = \Lambda_\nu^\mu a^\nu \quad (\text{contravariant transformation})$$

$$\bullet a'_\mu = (\Lambda^{-1})^\nu_\mu a_\nu \quad (\text{covariant transformation})$$

We start with:

$$\textcircled{1} \quad b^\rho = H^{\rho\sigma} a_\sigma$$

and

$$\textcircled{2} \quad b'^\mu = H'^{\mu\nu} a_\nu \Rightarrow \Lambda_\rho^\mu b^\rho = H'^{\mu\nu} (\Lambda^{-1})^\sigma_\nu a_\sigma$$

$$\Rightarrow b^\rho = (\Lambda_\rho^\mu)^{-1} H'^{\mu\nu} (\Lambda^{-1})^\sigma_\nu a_\sigma$$

$$(\text{comparing with eq } \textcircled{1}) \Rightarrow H^{\rho\sigma} = (\Lambda_\rho^\mu)^{-1} H'^{\mu\nu} (\Lambda^{-1})^\sigma_\nu$$

$$H'^{\mu\nu} = \Lambda_\rho^\mu H^{\rho\sigma} ((\Lambda^{-1})^\sigma_\nu)^{-1}$$

$$\Rightarrow \boxed{H'^{\mu\nu} = \Lambda_\rho^\mu H^{\rho\sigma} \Lambda^\nu_\sigma} \quad \text{QED}$$

$$\#6. \text{ Prove: } F'^{\mu\nu} = \Lambda_\rho^\mu F^{\rho\sigma} \Lambda^\nu_\sigma \quad \textcircled{1}$$

$$\text{where, } F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$(\text{starting with the LHS of eq } \textcircled{1}) \Rightarrow F'^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$(\text{Using transformations from problem \#5}) = (\Lambda_\sigma^\mu \partial^\sigma) (\Lambda_\rho^\nu A^\rho) - (\Lambda_\nu^\mu \partial^\nu) (\Lambda_\sigma^\rho A^\sigma)$$

$$(\text{Relabeling dummy indices}) = (\Lambda_\rho^\mu \partial^\rho) (A^\sigma \Lambda_\sigma^\nu) - (\Lambda_\nu^\mu \partial^\nu) (A^\sigma \Lambda_\sigma^\rho)$$

$$= \Lambda_\rho^\mu (\partial^\rho A^\nu - \partial^\nu A^\rho) \Lambda^\nu_\sigma$$

$$\Rightarrow \boxed{F'^{\mu\nu} = \Lambda_\rho^\mu F^{\rho\sigma} \Lambda^\nu_\sigma} \quad \text{QED}$$

$$\#7. F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

- $F^{00} = F^{11} = F^{22} = F^{33} = 0 \quad \Rightarrow \quad F^{00} = \partial^0 A^0 - \partial^0 A^0 = 0$
- $F^{0i} = \partial^0 A^i - \partial^i A^0 \quad (i=1,2,3)$

$$= \frac{\partial A^i}{c \partial t} - \frac{1}{c} \frac{\partial V}{\partial x^i}$$

So, $F^{0i} = E^i/c \Rightarrow F^{01} = \frac{E^1}{c} = E_x, F^{02} = \frac{E^2}{c} = E_y, F^{03} = \frac{E^3}{c} = E_z$

- $F^{i0} = \partial^i A^0 - \partial^0 A^i = -(\partial^0 A^i - \partial^i A^0) = -F^{0i}$

So, $F^{i0} = -E^i/c \Rightarrow F^{10} = -E^1/c, F^{20} = -E^2/c, F^{30} = -E^3/c$

- $F^{ij} = \partial^i A^j - \partial^j A^i \quad (i,j = 1,2,3)$

$$= \epsilon_{ijk} \partial^i A^j$$

$F^{ij} = B^k$
 and
 $F^{ji} = -B^k$

 $\Rightarrow F^{12} = B^3, F^{31} = B^2, F^{23} = B^1$
 $F^{21} = -B^3, F^{13} = -B^2, F^{32} = -B^1$

Thus,

$$F = \begin{pmatrix} 0 & E^1/c & -E^2/c & -E^3/c \\ E^1/c & 0 & -B^3 & B^2 \\ E^2/c & B^3 & 0 & -B^1 \\ E^3/c & -B^2 & B^1 & 0 \end{pmatrix}$$

$$\#8. \text{Prove: } \partial_\mu F^{\mu\nu} = \mu_0 J^\nu$$

$$\partial_\mu F^{\mu\nu} = \partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = \partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu$$

(using the result from problem #4(d)) $= \mu_0 J^\nu - \partial^\nu \cancel{\partial_\mu A^\mu}$ (Lorentz gauge condition problem #4 part (c)).

$$\Rightarrow \boxed{\partial_\mu F^{\mu\nu} = \mu_0 J^\nu} \quad \text{QED}$$

Continued...

Above we proved that the statement, $\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$, is true.

Now, let's show that this statement is equivalent to Maxwell's two source equations:

$$\partial_\mu F^{\mu\nu} = \partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu)$$

$$= \partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu$$

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = \mu_0 J^\nu$$

For $\nu = 0$:

$$\Rightarrow \partial_\mu \partial^\mu A^0 - \partial^0 \partial_\mu A^\mu = \mu_0 J^0$$

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) V_c - \frac{\partial}{\partial t} \left(\bar{\nabla} \cdot \bar{A} + \frac{1}{c^2} \frac{\partial V}{\partial t} \right) = \mu_0 \rho$$

$$\nabla^2 V + \bar{\nabla} \cdot \frac{\partial \bar{A}}{\partial t} = -\mu_0 c^2 \rho$$

$$\bar{\nabla} \cdot \left(\bar{\nabla} V + \frac{\partial \bar{A}}{\partial t} \right) = -\mu_0 \frac{1}{\mu_0 \epsilon_0} \rho$$

$\bar{\nabla} \cdot \bar{E} = \rho/\epsilon_0$

Maxwell's source equation

For $\nu = 1, 2, 3 = i$:

$$\Rightarrow \partial_\mu \partial^\mu A^i - \partial^i \partial_\mu A^\mu = \mu_0 J^i$$

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \bar{A} - (-\bar{\nabla}) \left(\bar{\nabla} \cdot \bar{A} + \frac{1}{c^2} \frac{\partial V}{\partial t} \right) = \mu_0 \bar{J}$$

$$(\bar{\nabla} \cdot (\bar{\nabla} \cdot \bar{A}) - \nabla^2 \bar{A}) + \frac{1}{c^2} \frac{\partial^2 \bar{A}}{\partial t^2} + \frac{1}{c^2} \frac{\partial \bar{\nabla} V}{\partial t} = \mu_0 \bar{J}$$

$$\bar{\nabla} \times (\bar{\nabla} \times \bar{A}) + \frac{1}{c^2} \frac{\partial}{\partial t} \left(\frac{\partial \bar{A}}{\partial t} + \bar{\nabla} V \right) = \mu_0 \bar{J}$$

$\bar{\nabla} \times \bar{B} = \mu_0 \bar{J} + \mu_0 \epsilon_0 \frac{\partial \bar{E}}{\partial t}$

Maxwell's source equation